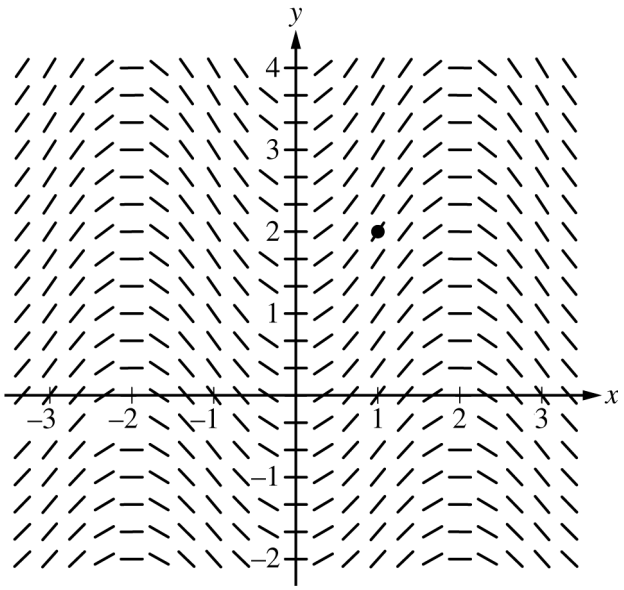
5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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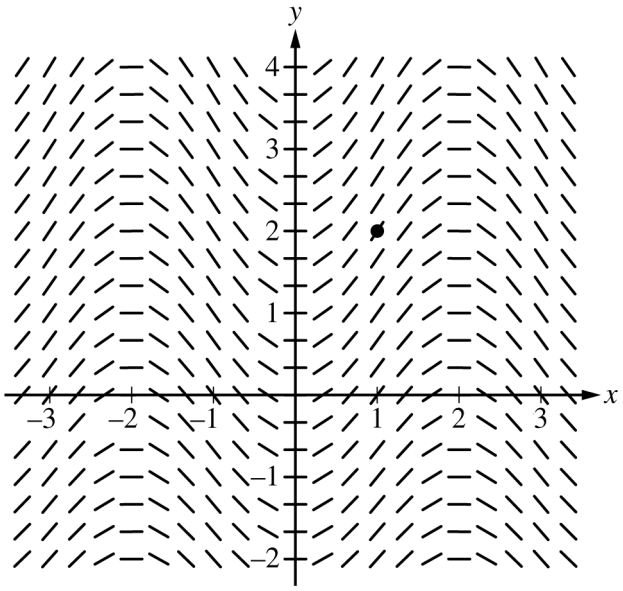
5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

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NO CALCULATOR ALLOWED

(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

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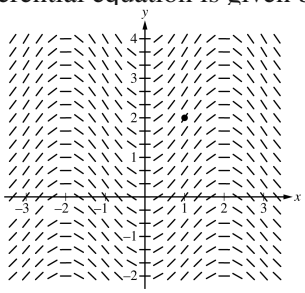
5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

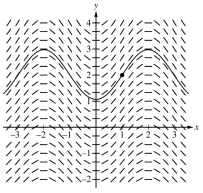
(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

Model Solution	Scoring
(a) 	1: Solution curve
(b) $\left.\frac{dy}{dx}\right\|_{(x,\,y)=(1,\,2)} = \frac{1}{2}\cdot 3\cdot\sin\left(\frac{\pi}{2}\right) = \frac{3}{2}$ An equation for the tangent line is $y = 2 + \frac{3}{2}(x-1)$. $f(0.8) \approx 2 + \frac{3}{2}(0.8-1) = 1.7$	2: {1: Tangent line equation; 1: Approximation}
(c) Because $f''(x) > 0$, f is concave up on $-1 \le x \le 1$, the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ is an underestimate.	1: Answer with reason
(d) $\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$ $2\sqrt{y+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right) + C$ $f(1) = 2 \Rightarrow 2\sqrt{2+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\cdot 1\right) + C$ $\Rightarrow 6 = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right) + C \Rightarrow C = 6$ $\sqrt{y+7} = 3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)$ $\quad y = \left(3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)\right)^2 - 7$	5: {1: separation; 2: Antiderivatives; 1: Constant of integration and uses intial condition; 1: Solves for y}

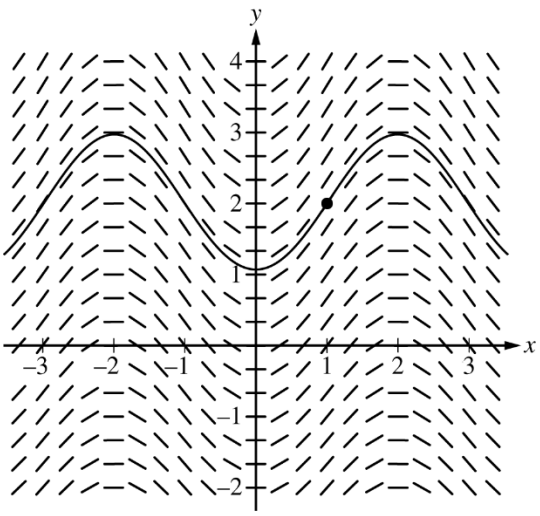
Part B (AB): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

	Model Solution	Scoring
(a)	A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.	
		Solution curve **1 point**

Scoring notes:

- The solution curve must pass through the point $(1, 2)$, extend reasonably close to the left and right edges of the square, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must indicate $f(x) > 0$ for all points on the curve.
- All local maximum/minimum points on the solution curve must occur at horizontal line segments in the slope field.

Total for part (a) 1 point

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(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

$\left.\frac{dy}{dx}\right\|_{(x, y)=(1, 2)} = \frac{1}{2} \cdot 3 \cdot \sin\left(\frac{\pi}{2}\right) = \frac{3}{2}$ An equation for the tangent line is $y = 2 + \frac{3}{2}(x - 1)$.	Tangent line equation	**1 point**
$f(0.8) \approx 2 + \frac{3}{2}(0.8 - 1) = 1.7$	Approximation	**1 point**

Scoring notes:

- The tangent line equation can be presented in any equivalent form.
- An incorrect tangent line equation with a slope of $\frac{3}{2}$ is eligible to earn the second point for a consistent answer.
- A response of only $2 + \frac{3}{2}(0.8 - 1)$ earns the second point but not the first.

Total for part (b) 2 points

(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Because $f''(x) > 0$, f is concave up on $-1 \le x \le 1$, the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ is an underestimate.	Answer with reason	**1 point**

Scoring notes:

- The reason must include $f''(x) > 0$, $f'(x)$ is increasing, or $f(x)$ is concave up.

Total for part (c) 1 point

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(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

$\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$	Separation of variables	**1 point**
$2\sqrt{y+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right) + C$	One correct antiderivative	**1 point**
	The other correct antiderivative	**1 point**
$f(1) = 2 \Rightarrow 2\sqrt{2+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\cdot 1\right) + C$ $\Rightarrow 6 = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right) + C \Rightarrow C = 6$ $\sqrt{y+7} = 3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)$	Constant of integration and uses initial condition	**1 point**
$y = \left(3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)\right)^2 - 7$	Solves for y	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points.
- A response with no constant of integration can earn at most the first 3 points.
- A response is eligible for the fourth point only if it has earned the first point and at least 1 of the 2 antiderivative points.
  - Special case: The incorrect separation of $\sqrt{y+7}\ dy = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$ does not earn the first point, is only eligible for the antiderivative point for $-\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)$, and is eligible for the fourth point.
- An eligible response earns the fourth point by correctly including the constant of integration in an equation and substituting 1 for x and 2 for y.
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (d) 5 points

Total for question 5 9 points

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